

BOUNDS ON EXCEPTIONAL DEHN FILLING II

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ABSTRACT. We show that there are at most finitely many one cusped orientable hyperbolic 3-manifolds which have more than eight non-hyperbolic Dehn fillings. Moreover, we show that determining these finitely many manifolds is decidable.

1. INTRODUCTION

Thurston demonstrated that if one has a hyperbolic knot complement, all but finitely many Dehn fillings give hyperbolic manifolds [38]. The example with the largest known number of non-hyperbolic Dehn fillings is the figure-eight knot complement, which has 10 fillings which are not hyperbolic. It is conjectured that this is the maximal number that can occur. Call a Dehn filling *exceptional* if it is not hyperbolic. Previous authors have distinguished between hyperbolike manifolds, which are irreducible, atoroidal, with infinite fundamental group, and hyperbolic manifolds, defining an exceptional filling to be one which is not hyperbolike. But by the geometrization theorem [34, 33, 25, 29, 13], a manifold is hyperbolic if and only if it is hyperbolike, so we need no longer make this distinction. Bleiler and Hodgson [6] showed that there are at most 24 exceptional Dehn fillings, using Gromov and Thurston's 2π -theorem and estimates on cusp size due to Colin Adams [1]. We made an improvement on the 2π -theorem [2], independently discovered by Lackenby [26], and used improved lower bounds on cusp size due to Cao and Meyerhoff [12], to get an upper bound of 12 exceptional Dehn fillings. In this paper, we show that there are at most finitely many one cusped hyperbolic manifolds which have exceptional Dehn fillings r_1, r_2 such that $\Delta(r_1, r_2) > 5$. This theorem is sharp, in that the $(-2, 3, 8)$ pretzel link complement W' admits two exceptional Dehn fillings r_1, r_2 with $\Delta(r_1, r_2) = 5$. By hyperbolic Dehn filling the other cusp of W' , we see that there are infinitely many 3-manifolds with $\Delta(r_1, r_2) = 5$. This theorem implies that there are only finitely many one cusped hyperbolic 3-manifolds with > 8 exceptional Dehn fillings, since there can be at most 8 curves on a torus with $\Delta(r_1, r_2) \leq 5$. We also prove the existence of an algorithm which will determine the manifolds which have $\Delta(r_1, r_2) > 5$ for exceptional Dehn fillings r_1, r_2 , and therefore which manifolds may have > 8 exceptional Dehn fillings. It is conjectured that there are only four orientable hyperbolic manifolds with two exceptional Dehn fillings r_1, r_2 so that $\Delta(r_1, r_2) > 5$. These manifolds are the Dehn fillings $W(-1), W(5), W(5/2)$, and $W(-2)$ on one component of the Whitehead link complement W , and are conjectured by Gordon to be the only such examples [20, Conjecture 3.4]. Of these four examples, only the figure eight knot complement $W(-1)$ has > 8 exceptional Dehn fillings. It is conjectured that there are only finitely many one cusped hyperbolic 3-manifolds with > 6 exceptional Dehn fillings [24, Problem 1.77].

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Our theorem depends on results of Anderson, Canary, Culler and Shalen [5] which have been subsequently strengthened by a culmination of various results in Kleinian groups. In particular, we use Ohshika's result that Kleinian groups are limits of geometrically finite Kleinian groups [31] (see also [30]), which depends on the classification of Kleinian groups, including the tameness conjecture [3, 11] (see also [37]) and the ending lamination conjecture [9, 28], and generalizes many previous results on density of geometrically finite Kleinian groups in the space of Kleinian groups [10, 8], going back to work of Jorgensen on the space of punctured torus groups [23] and Thurston's double limit theorem [39, 32]. We remark that Ohshika's argument [31] depends on the ending lamination conjecture for general Kleinian groups, which has been claimed by Brock, Canary and Minsky and disseminated through talks at various conferences, but for which there is as yet no preprint. The preprints [9, 28] only treat the case of freely-indecomposable Kleinian groups, but we need the result for freely decomposable groups for the application in this paper. The key geometric consequence that we make use of is the following result stated in Theorem 3.2: if a maximal horocusp in an orientable hyperbolic manifold has volume $< \pi - \epsilon$ for $\epsilon > 0$, then the volume of the manifold is uniformly bounded as a function of ϵ .

2. BACKGROUND AND DEFINITIONS

Let N be a hyperbolic 3-manifold. Then $N \cong \mathbb{H}^3/\Gamma$ where $\Gamma \cong \pi_1(N)$ is a discrete torsion-free subgroup of $\text{Isom}(\mathbb{H}^3)$, with $p : \mathbb{H}^3 \rightarrow N$ the covering map. We will assume that the reader is familiar with Margulis' constant ϵ and the thick-thin decomposition in the case of hyperbolic 3-manifolds; see [38, Chapter 5] for an introduction to this. The following definitions are needed only for part of the proof of Lemma 3.1. If g is a loxodromic isometry of hyperbolic 3-space \mathbb{H}^3 , we shall let A_g denote the hyperbolic geodesic which is the axis of g . The cylinder about A_g of radius r is the open set $Z_r(g) = \{x \in \mathbb{H}^3 \mid \text{dist}(x, A_g) < r\}$. If C is a simple closed geodesic in N then there is a primitive loxodromic isometry $g \in \Gamma$ with $p(A_g/\langle g \rangle) = C$. For any $r > 0$, the projection $p(Z_r(g)/\langle g \rangle)$ of $Z_r(g)$ under the covering projection is a neighborhood of C in N . For sufficiently small $r > 0$ we have $\{h \in \Gamma \mid h(Z_r(g)) \cap Z_r(g) \neq \emptyset\} = \langle g \rangle$. Let R denote the supremum of the set of r for which this condition holds. We define $\text{tube}(C) = Z_R(g)/\langle g \rangle$ to be the maximal tube about C .

There is a similar situation when we have a maximal parabolic subgroup $P < \Gamma$. We define an open horoball to be a subset of \mathbb{H}^3 isometric to $\{(z, t) \mid z \in \mathbb{C}, t > 1\}$, where $\mathbb{H}^3 = \{(z, t) \mid t > 0\} \subset \mathbb{C} \times \mathbb{R}$ is the upper half-space model of hyperbolic 3-space. There is an open horoball $\tilde{\mathcal{H}} \subset \mathbb{H}^3$ so that $P = \{g \in \Gamma \mid g(\tilde{\mathcal{H}}) = \tilde{\mathcal{H}}\}$. Choose $\tilde{\mathcal{H}}$ maximal subject to the condition that $\{h \in \Gamma \mid h(\tilde{\mathcal{H}}) \cap \tilde{\mathcal{H}} \neq \emptyset\} = P$. Then we call $p(\tilde{\mathcal{H}}) = \mathcal{H} \cong \tilde{\mathcal{H}}/P$ a *maximal horocusp*. If $P \cong \mathbb{Z}^2$, then $\mathcal{H} \cong S^1 \times S^1 \times \mathbb{R}$, and if $P \cong \mathbb{Z}$, then $\mathcal{H} \cong S^1 \times \mathbb{R}^2$. Since \mathcal{H} is maximal subject to this condition, there exists $\gamma \in \Gamma - P$ so that $\gamma(\partial\tilde{\mathcal{H}}) \cap \partial\tilde{\mathcal{H}} = x$. Note that $\gamma^{-1}(\partial\tilde{\mathcal{H}}) \cap \partial\tilde{\mathcal{H}} = \gamma^{-1}(x)$ will also be a point of tangency. Then $\gamma^{-1}(\tilde{\mathcal{H}})$ is also a horoball of height one call the Adam's horoball [1] (see Figure 1). We will call the group $B = \langle P, \gamma \rangle$ a *bicuspid* group, so called because it corresponds to a point in \mathbb{H}^3 where the closure of two maximal horoballs meet, corresponding to two preimages of a neighborhood of a cusp of N . In this paper, when $\mathbb{Z}^2 \cong P$, we will use the notation $P = \langle \alpha, \beta \rangle$, where α, β are parabolic elements. Then B is *free bicuspid* if $B \cong P * \langle \gamma \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$. For

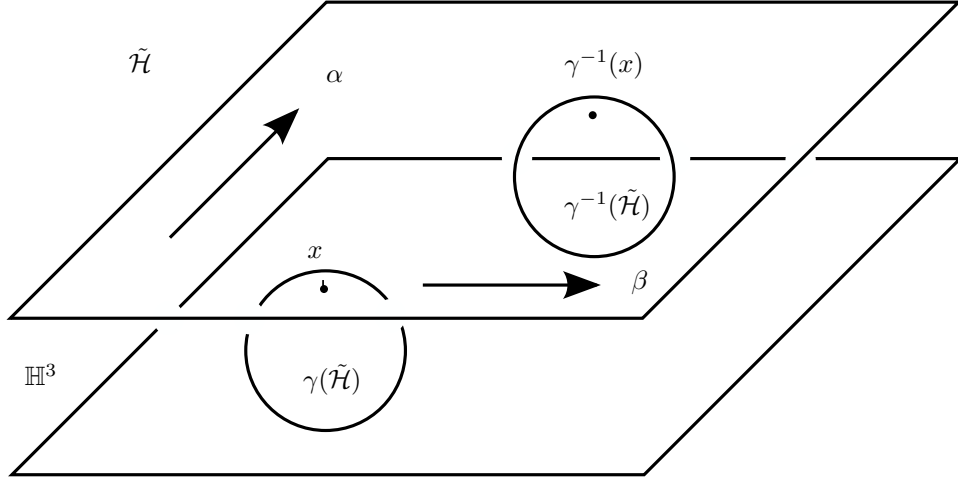


FIGURE 1. A bicuspid group associated to a maximal horocusp

any cusp of N with associated maximal horocusp \mathcal{H} , there will be finitely many bicuspid subgroups of Γ up to conjugacy in one-to-one correspondence with the Γ orbits of points $x \in \mathbb{H}^3$ so that $x = \gamma(\partial\tilde{\mathcal{H}}) \cap \partial\tilde{\mathcal{H}}$ for some $\gamma \in \Gamma - P$. One may think of each of these points as “self-tangencies” of $\partial\mathcal{H}$.

3. MAXIMAL EMBEDDED CUSPS OF HYPERBOLIC MANIFOLDS

The following geometric result is made possible by recent advances in Kleinian groups, and generalizes [4, Lemma 4.3].

Lemma 3.1. *Let $\langle \alpha, \beta, \gamma \rangle$ be a free bicuspid group, with maximal horocusp \mathcal{H} . Then $\text{Vol}(\mathcal{H}) \geq \pi$.*

Proof. Let $N = \mathbb{H}^3 / \langle \alpha, \beta, \gamma \rangle$. Since \mathcal{H} is maximal, there is a point of self-contact on $\partial\mathcal{H}$. Let $\mathbb{Z} + \mathbb{Z} \cong \pi_1(\mathcal{H}) = \langle \alpha, \beta \rangle < \pi_1(N)$. Since the cusp \mathcal{H} is maximal, there is a component of the preimage of \mathcal{H} in the universal cover \mathbb{H}^3 fixed by covering translations $\langle \alpha, \beta \rangle$ such that $\gamma(\partial\tilde{\mathcal{H}}) \cap \partial\tilde{\mathcal{H}} \neq \emptyset$. Denote $Q = \langle \alpha, \beta, \gamma \rangle = \pi_1(N)$. We have $\text{Area}(\partial\mathcal{H}) = \frac{1}{2}\text{Vol}(\mathcal{H})$. For sake of contradiction, we may assume that $\text{Area}(\partial\mathcal{H}) \leq 2\pi$. Choose α to represent the shortest path in $\partial\mathcal{H}$, and β the next shortest path. By [1], the length of a geodesic path representing α in $\partial\mathcal{H}$ is ≥ 1 . We may normalize α, β, γ as elements in $\text{SL}_2\mathbb{C}$ up to conjugacy, such that $\tilde{\mathcal{H}}$ is the horoball centered at ∞ of height 1, and

$$\alpha = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \gamma = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix},$$

where $1 \leq |a| \leq |b| \leq 4\pi/\sqrt{3}$, and $|c| \leq |b|$.

To prove the theorem, first we approximate Q by geometrically finite groups Q_i such that $Q_i = \langle \alpha_i, \beta_i, \gamma_i \rangle$ and such that $\alpha_i \rightarrow \alpha, \beta_i \rightarrow \beta, \gamma_i \rightarrow \gamma$. Denote by $N_i = \mathbb{H}^3 / Q_i$ the hyperbolic manifold which is homeomorphic to the interior of a boundary connect sum $(T^2 \times I) \amalg (S^1 \times D^2) \cong \overline{N}_i$. Denote S_i to be the torus boundary component of \overline{N}_i . In fact, one may assume that in the groups

Q_i , the only parabolic elements are conjugate into $\langle \alpha_i, \beta_i \rangle$. This is possible by the fact that geometrically finite groups without rank one parabolics are dense in the space of all Kleinian groups in the algebraic topology [31, Theorem 1.1]. We may assume that N_i contains a maximal embedded horocusp neighborhood \mathcal{H}_i such that $\text{Vol}(\mathcal{H}_i) \rightarrow \text{Vol}(\mathcal{H})$.

Next, we proceed as in Lemma 4.3 of [4]. Suppose that (P_j) is an infinite sequence of distinct hyperbolic manifolds obtained by Dehn filling N_i along S_i using Thurston's hyperbolic Dehn surgery theorem for geometrically finite manifolds (see [7] or [14]). Then $\pi_1(P_j)$ is free two generator, and the manifolds (P_j) converge geometrically to N_i . Moreover, the core curve of the Dehn filling P_j of N_i is isotopic to a geodesic C_j in P_j . The length L_j of C_j tends to 0 as $j \rightarrow \infty$; and the sequence of maximal tubes $(\text{tube}(C_j)), j \geq 1$ converges geometrically to \mathcal{H}_i . In particular $\lim_{j \rightarrow \infty} \text{Vol}(\text{tube}(C_j)) = \text{Vol}(\mathcal{H}_i)$. According to [4, Corollary 4.2], $\log 3$ is a strong Margulis number for each of the hyperbolic manifolds P_j . It therefore follows from [5, Corollary 10.5] that $\text{Vol}(\text{tube}(C_j)) > V(L_j)$, where V is an explicitly defined function such that $\lim_{x \rightarrow 0} V(x) = \pi$. In particular, this shows that $\text{Vol}(\mathcal{H}_i) \geq \lim_{j \rightarrow \infty} V(L_j) \geq \pi$. Now, since \mathcal{H}_i converges geometrically to \mathcal{H} , we conclude that $\text{Vol}(\mathcal{H}) \geq \pi$. \square

Remark: The estimates of [5] depend on a paradoxical decomposition argument for a 2-generator free group acting on \mathbb{H}^3 given in [16]. We believe that it should be possible to give a more direct argument for the previous lemma by analyzing an appropriate generalization of the paradoxical decomposition for free bicuspid groups.

Example: Consider the group $\Gamma = \langle \alpha, \beta, \gamma \rangle$, where

$$\alpha = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 1+i\sqrt{3} \\ 0 & 1 \end{pmatrix}, \gamma = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

One may show that this group is free. Consider the geodesic planes in the upper half space model of \mathbb{H}^3 bounding the circles $\{|z - c| = 2 \mid c \in \mathbb{Z}2 + \mathbb{Z}(1 + i\sqrt{3})\}$, and cut out the open half-spaces disjoint from ∞ bounded by these planes to obtain a region $R \subset \mathbb{H}^3$. These bounding circles have two orbits under the group $\langle \alpha, \beta \rangle$, with representatives at centers $c = 0, 2$, and thus $R/\langle \alpha, \beta \rangle$ will be homeomorphic to $T^2 \times \mathbb{R}$ with two geodesic disks in its boundary. Under the map γ , the circle $|z| = 2$ is sent to the circle $|z - 2| = 2$, and therefore γ maps the corresponding planes bounding these circles to themselves. Thus, $N = \mathbb{H}^3/\Gamma$ is obtained from $R/\langle \alpha, \beta \rangle$ by gluing the two geodesic planes in the the boundary using the identification given by γ . This has the effect of adding a handle onto $T^2 \times \mathbb{R}$, and therefore N is homeomorphic to the interior of a compression body with fundamental group identified with $\Gamma \cong \langle \alpha, \beta \rangle * \langle \gamma \rangle$. Thus, we see that Γ is free bicuspid. If $\tilde{\mathcal{H}}$ is the horosphere centered at ∞ of height one in \mathbb{H}^3 , then $\tilde{\mathcal{H}}/\langle \alpha, \beta \rangle$ embeds as a cusp $\mathcal{H} \subset N$, since it is disjoint from the geodesic planes given above. We have $\text{Vol}(\mathcal{H}) = 2\sqrt{3} = 3.46\dots$. Thus, the bound given in Lemma 3.1 is fairly close to optimal.

We did a search of manifolds in the Snappea census [40], and found that the manifold $M = \text{v1902}$ has an embedded cusp of volume $3.238\dots$. Moreover, the cusp group associated to v1902 is free. One may see this by finding an irregular 3-fold cover $\tilde{M} \rightarrow M$ (cover 8 in Snappea's notation) which has 3 maximal cusps

with the same volume as M , and has $H_1(\tilde{M}) = \mathbb{Z}^3 + \mathbb{Z}/3\mathbb{Z}$ (see Figure 2). It follows that the bicuspid subgroup of $\pi_1 M$ is free, since it lifts to $\pi_1 \tilde{M}$, and any bicuspid subgroup of $\pi_1 \tilde{M}$ is free since $\text{rank} H_1(\tilde{M}; \mathbb{Z}/3\mathbb{Z}) = 4$ (see the argument of [4, Prop. 5.3]).

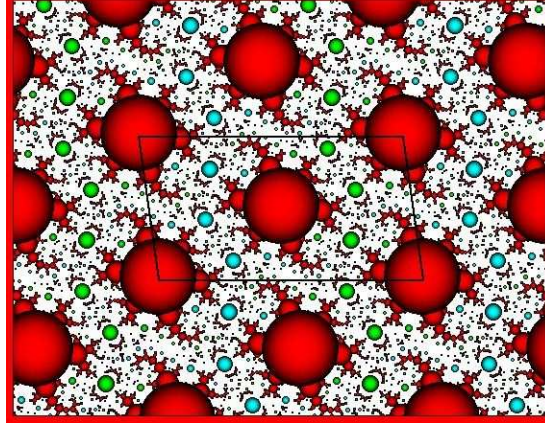


FIGURE 2. Pattern of horoballs in \mathbb{H}^3 for the manifold v1902 3-fold cover 8 from the Snappea census with cusp volume 3.238...

In fact, there will be an optimal constant $\pi \leq R < 3.238\dots$ such that the quotient manifold of every free bicuspid group will have an embedded horocusp \mathcal{H} with $\text{Vol}(\mathcal{H}) \geq R$. It is an interesting question to obtain better upper and lower bounds on R . The following theorem would hold with R replacing π .

Theorem 3.2. *For $\epsilon > 0$, there exists a constant $V(\epsilon)$ such that if N is an orientable hyperbolic 3-manifold with a maximal horocusp \mathcal{H} with $\text{Vol}(\mathcal{H}) < \pi - \epsilon$, then $\text{Vol}(N) < V(\epsilon)$.*

Proof. Let $Q < \pi_1 N$ be a bicuspid subgroup corresponding to the maximal horocusp \mathcal{H} . First, suppose that Q is a free product $Q = \langle \alpha_1, \alpha_2 \rangle * \langle \beta \rangle \cong (\mathbb{Z} + \mathbb{Z}) * \mathbb{Z}$. Then $\text{Vol}(\mathcal{H}) > \pi$ by Lemma 3.1.

Next, suppose Q is not a free product. Then Q must be indecomposable, and in fact \mathbb{H}^3/Q must be finite volume. We get a contradiction in this case by taking a geometric limit. Suppose we have $\Gamma_j < \text{PSL}(2, \mathbb{C})$ with $N_j = \mathbb{H}^3/\Gamma_j$, such that $\text{Vol}(N_j) \rightarrow \infty$ and maximal bicuspid subgroups $Q_j < \Gamma_j$, such that Q_j is not free bicuspid. Then $\text{Vol}(\mathbb{H}^3/Q_j) < \infty$ by [22, Theorem IV.4.1], so $[\Gamma_j : Q_j] < \infty$. We may assume that $\text{Vol}(\mathbb{H}^3/Q_j) \rightarrow \infty$ as $j \rightarrow \infty$, since otherwise there exists V such that $V > \text{Vol}(\mathbb{H}^3/Q_j) \geq \text{Vol}(\mathbb{H}^3/\Gamma_j)$, which is what we wanted to conclude. Normalize Q_j as a standard bicuspid group with $\mathcal{H}_j \subset \mathbb{H}^3/Q_j$ the maximal cusp, and with self-tangency point of $x_j \in \partial \mathcal{H}_j \subset \mathbb{H}^3/Q_j$. Take a subsequence j_k such that $(\mathbb{H}^3/Q_{j_k}, x_{j_k})$ converges geometrically to (M_∞, x_∞) , where $\mathcal{H}_\infty \subset M_\infty$ is a maximal cusp with self-tangency point x_∞ and $\text{Vol}(M_\infty) = \infty$. Then Q_j converges algebraically to a bicuspid subgroup $Q_\infty < \pi_1(M_\infty)$ such that \mathcal{H}_∞ lifts to a maximal cusp in \mathbb{H}^3/Q_∞ . By [22, Theorem IV.4.1], $Q_\infty \cong (\mathbb{Z} + \mathbb{Z}) * \mathbb{Z}$, so that Q_∞ is a free bicuspid group. The cusps $\mathcal{H}_j \rightarrow \mathcal{H}_\infty$ in the Gromov-Hausdorff topology. Since $\text{Vol}(\mathcal{H}_\infty) \geq \pi$ by Lemma 3.1, we see that there is k such that $\text{Vol}(\mathcal{H}_{j_k}) > \pi - \epsilon$.

This gives a contradiction, so we see that the assumption that $\text{Vol}(\mathbb{H}^3/Q_j) \rightarrow \infty$ is false, and thus there exists $V(\epsilon)$ such that $\text{Vol}(N_j) < V(\epsilon)$. \square

Remark: A similar argument to the previous theorem shows that there is a constant V such that if M is a hyperbolic 3-manifold with Margulis constant $< \log 3$, then $\text{Vol}(M) < V$ by applying [16]. It would be interesting to get some idea of the distribution of Margulis constants $< \log 3$.

4. EXCEPTIONAL DEHN FILLINGS

In this section, we prove the main theorem of the paper.

Theorem 4.1. *There are only finitely many one cusped orientable hyperbolic 3-manifolds of finite volume N such that N has two exceptional Dehn fillings a_1, a_2 so that $\Delta(a_1, a_2) > 5$.*

Proof. Fix some $\epsilon < \pi - 3$. If $\text{Vol}(N) > V(\epsilon)$, where $V(\epsilon)$ is the constant from Theorem 3.2, then N has a maximal horocusp $\mathcal{H} \subset N$ so that $\text{Vol}(\mathcal{H}) \geq \pi - \epsilon > 3$. Then $\text{Area}(\partial\mathcal{H}) > 6$. By the proof of [2, Theorem 8.1], if a_1, a_2 are two slopes on $\partial\overline{N}$ so that $\overline{N}(a_i)$ are exceptional Dehn fillings, then $\Delta(a_1, a_2) \leq 6^2/\text{Area}(\partial\mathcal{H}) < 6$. Thus, N does not violate the theorem.

Suppose the theorem is false. Then there is an infinite sequence of orientable one cusped hyperbolic 3-manifolds $(N_i)_{i \in \mathbb{N}}$ such that $i \rightarrow \infty$ and N_i has two exceptional Dehn fillings of distance > 5 . By the previous paragraph, we may assume that $\text{Vol}(N_i) < V(\epsilon)$. Let \mathcal{H}_i be a maximal cusp neighborhood in N_i . By [38, Theorem 5.12.1], we may choose a subsequence $J \subset \mathbb{N}$ and an orientable hyperbolic 3-manifold M with ≥ 2 rank two cusps so that each $N_i, i \in J$ is obtained by hyperbolic Dehn filling on M , and therefore N_i converges to M in the Gromov-Hausdorff topology. Moreover, M has a distinguished cusp with maximal cusp neighborhood \mathcal{H} such that as $i \rightarrow \infty, i \in J$, $\mathcal{H}_i \rightarrow \mathcal{H}$ in the Gromov-Hausdorff topology (in fact, since $\text{Vol}(\mathcal{H}) \leq \pi - \epsilon$, we may assume that M has at most three cusps). By [19, Theorem 1.3], the distance between exceptional filling slopes on the cusp \mathcal{H} of M is ≤ 5 . Since $N_i \rightarrow M$ as $i \rightarrow \infty, i \in J$, it follows that there is a subset $J_0 \subset J$, $|J - J_0| < \infty$, so that the distance between filling slopes on N_i for $i \in J_0$ is ≤ 5 . Let $\partial\overline{M} = T_0 \cup T_1 \cup T_2$, where T_0 corresponds to \mathcal{H} . For slopes $q_j \in \mathbb{Q} \cup \infty \cup *$, $j = 0, 1, 2$, let $M(q_0, q_1, q_2)$ be Dehn filling on the boundary component T_i with slope q_i , where the $*$ denotes that the boundary component is unfilled. Let $\overline{N}_i = M(*, q_{i,1}, q_{i,2})$, $i \in J$. For each $0 \leq j \leq 2$, there is a finite subset $E_j \subset \mathbb{Q} \cup \infty$ so that if $q_j \notin E_j$, then $M(q_0, q_1, q_2)$ is hyperbolic [38, Theorem 5.8.2]. For $i \in J$ large enough, $q_{i,j} \notin E_j$, for $j = 1, 2$. This implies that the only non-hyperbolic Dehn fillings on N_i must correspond to a subset of the slopes E_0 . Let $E_h \subset E_0$ be such that $t \in E_h$ if and only if $M(t, *, *)$ is hyperbolic. For $t_1, t_2 \in E_0 - E_h$, we have $\Delta(t_1, t_2) \leq 5$ by [19, Theorem 1.3] as noted above. For each slope $t \in E_h$ there are subsets $E_{t,j} \subset \mathbb{Q} \cup \infty$ so that $M(t, q_1, q_2)$ is hyperbolic if $q_j \notin E_{t,j}$ (again by [38, Theorem 5.8.2]). Let $F_j = E_j \cup_{t \in E_h} E_{t,j}$. Then if $q_{i,j} \notin F_j$ and $t \in E_h$, then $N_i(t)$ is also hyperbolic. Therefore, we see that for i large, the distance between two exceptional slopes of N_i is at most 5. This gives a contradiction to our assumption that there is a subsequence of manifolds contradicting the theorem with volumes bounded by $V(\epsilon)$, thus proving the theorem. \square

Corollary 4.2. *There are only finitely many one cusped orientable hyperbolic 3-manifolds of finite volume N such that N has > 8 exceptional Dehn fillings.*

Proof. By [2, Lemma 8.2], if the distance between exceptional Dehn fillings is ≤ 5 , take the next largest prime and add one, to conclude there are at most 8 exceptional Dehn fillings on N . \square

5. AN ALGORITHM TO FIND EXCEPTIONAL DEHN FILLINGS

We would like to classify the finitely many manifolds with two exceptional Dehn fillings of distance > 5 given by Theorem 4.1. Since the proof of Theorem 3.2 is by contradiction, it's not clear that there is a procedure one could run which would for a given ϵ identify all of the manifolds which have an embedded cusp of volume $< \pi - \epsilon$. Thus, we must devise a method to classify manifolds with a small volume cusp. This is similar to the procedures implemented in [18, 17], which find manifolds with small radius tubes around a short geodesic, or small volume cusped manifolds. We modify their approach to show that finding the manifolds with exceptional fillings in Theorem 4.1 and Corollary 4.2 is decidable. First, we need some preliminary results.

Theorem 5.1. *Given a finite volume hyperbolic manifold M , there is an algorithm which will determine the set of all exceptional Dehn fillings on M for which any proper sub-Dehn filling is hyperbolic.*

Proof. We approximate the hyperbolic structure on the cusped manifold well enough to determine the rough shape (for example using interval arithmetic) of simultaneously embedded equal sized horocusp neighborhoods of all the cusps. This uses the algorithms to compute Ford domains in [36, 21, 41] (see also [27]). We then apply [2, 26] to determine which slopes on each cusp have length ≤ 6 . We perform all of the Dehn fillings along these short slopes, determining which ones result in hyperbolic manifolds using the algorithms described in [36, 41, 27], then repeat. At each stage we get hyperbolic manifolds with fewer cusps, and so this process eventually terminates with a finite collection of cusped hyperbolic manifolds along with a finite collection of slopes associated to each cusp, so that every exceptional Dehn filling on M is obtained by Dehn filling on one of the slopes associated to one of the members of this finite collection. \square

Lemma 5.2. *Let N be a hyperbolic 3-manifold, and $R \subset N$ be a connected region, such that $\text{im}\{\pi_1(R) \rightarrow \pi_1(N)\}$ is elementary. Then $\text{Vol}(R) \leq \frac{1}{2}\text{Area}(\partial R)$.*

Proof. This follows from the isoperimetric inequality for \mathbb{H}^3 plus the fact that the image of $\pi_1 R$ in $\pi_1 N$ is amenable. Alternatively, one may also use the method of [35, Lemma 3.2, Theorem 4.1], which is essentially a calibration argument. \square

Lemma 5.3. *Suppose $Q = \langle \alpha, \beta, \gamma \rangle$ is a (discrete torsion-free) bicuspid group which is not free. Let $N = \mathbb{H}^3/Q$. Let $w(x, y, z) \in (\langle x \rangle \times \langle y \rangle) * \langle z \rangle$ be a cyclically reduced word, such that $w(\alpha, \beta, \gamma) = 1 \in Q$. Let $d(w)$ be the number of occurrences of $z^{\pm 1}$ in the word w . Then $\text{Vol}(\mathbb{H}^3/Q) \leq \pi(d(w) - 2)$.*

Proof. This result generalizes [15]. Consider the word $w = r_1 z^{\epsilon_1} r_2 z^{\epsilon_2} \dots r_k z^{\epsilon_k}$, where we assume that $\epsilon_i \neq 0$, and $r_i \in \langle x \rangle \times \langle y \rangle - \{1\}$. Then $d(w) = \sum_{i=1}^k |\epsilon_i|$. For $k = 0, \dots, d(w)$, let $w_k(x, y, z)$ be a prefix of w so that $d(w_k) = k$ and w_k ends in $z^{\pm 1}$ (thus $w_{d(w)} = w$). Then we may find a string of geodesics γ_k , $k = 1, \dots, d(w)$

in \mathbb{H}^3 so that γ_k connects $w_{k-1}(\alpha, \beta, \gamma)(\infty)$ and $w_k(\alpha, \beta, \gamma)(\infty)$, and so that $\gamma_{d(w)}$ connects $w_{d(w)-1}(\alpha, \beta, \gamma)(\infty)$ to $\infty = w(\infty)$. We may find a map of a disk $r : D \rightarrow \mathbb{H}^3$ with $d(w)$ punctures in ∂D so that $r(\partial D) \subset \gamma_1 \cup \dots \cup \gamma_{d(w)}$ by coning each γ_i to ∞ , for $1 < i < d(w)$. The disk $r(D)$ will be made of $d(w) - 2$ triangles, and therefore $\text{Area}(r(D)) = \pi(d(w) - 2)$. Under the projection map $p : \mathbb{H}^3 \rightarrow N$, each γ_i will project to the same geodesic $\gamma \subset N$, and the boundary projects to $p \circ r(\partial D) \subset \gamma$.

Consider a component $U \subset N \setminus p \circ r(D)$, and let $W \subset U$ be a compact submanifold with boundary which is a deformation retract of U . For each sphere $S \subset N \setminus W$, there is a ball $B \subset N$, $\partial B = S$. If $W \subset B$ for some ball B , then $\text{im}\{\pi_1 W \rightarrow \pi_1 N\} = \text{im}\{\pi_1 U \rightarrow \pi_1 N\} = 1$. Otherwise, $B \subset N \setminus W$ for each 2-sphere $S \subset N \setminus W$, and therefore $N \setminus W$ is irreducible. If $T^2 \subset N \setminus W$ is a torus, then either T^2 is incompressible in N , and therefore $T = \partial V$, where V is a neighborhood of a cusp end of N , or T is compressible in N . If T is compressible in N , then $T = \partial V$, where V is a ball with knotted hole (possibly a solid torus). In either case, we see that if $W \subset V$, then $\text{im}\{\pi_1 U \rightarrow \pi_1 N\}$ is contained in an elementary subgroup. So if $\text{im}\{\pi_1 U \rightarrow \pi_1 N\}$ is not contained in an elementary subgroup, we conclude that $N \setminus W$ is irreducible and atoroidal. This implies that some component $\Sigma \subset \partial W$ must have $\chi(\Sigma) < 0$, since any sphere or torus boundary component of ∂W would bound a ball or ball with knotted hole component or cusp neighborhood of $N \setminus W$, which would contradict the fact that $N \setminus U \supset p(r(D))$ is connected. Therefore, $\chi(N \setminus W) < 0$. Let $G = \text{im}\{Q \rightarrow \pi_1(N \setminus W)\}$, where we identify Q with the image of $\pi_1(\gamma \cup \mathcal{H})$, where $\mathcal{H} \subset N \setminus W$ is a horocusp. We then have G must be a free bicuspid subgroup of $\pi_1(N \setminus W)$, since $\chi(N \setminus W) < 0$ by [22, Theorem IV.4.1]. But by hypothesis, $w(x, y, z)$ gives a non-trivial relation for G , since $p(r(D)) \subset N \setminus W$. This gives a contradiction, unless for each component U of $N \setminus r(p(D))$, we have $\text{im}\{\pi_1 U \rightarrow \pi_1 N\}$ lies in an elementary subgroup. Therefore, we conclude that $\text{Vol}(U) \leq \frac{1}{2} \text{Area}(\partial U)$ by Lemma 5.2. We conclude that $\text{Vol}(N) = \text{Vol}(N \setminus r(p(D))) \leq \text{Area}(r(p(D))) \leq \pi(d(w) - 2)$. \square

Theorem 5.4. *Given a rational number V , there is an algorithm which will find a finite collection M_1, \dots, M_m of finite volume orientable hyperbolic 3-manifolds so that any hyperbolic 3-manifold of volume $< V$ is obtained by Dehn filling on one of the manifolds M_i .*

Proof. Let ε be Margulis' constant for hyperbolic 3-manifolds. By the method of proof of the Jorgensen-Thurston theorem, there is a constant C so that if M is a hyperbolic 3-manifold with $\text{Vol}(M) < V$, then $M_{\text{thick}(\varepsilon)}$ admits a triangulation with $< CV$ tetrahedra. The first step of the algorithm is to take $< CV$ tetrahedra, and glue them together in all possible ways to get an orientable manifold with Euler characteristic zero. Next, we run the algorithm described in [21, 41, 27] to decide which of these manifolds has a hyperbolic interior. Then all hyperbolic 3-manifolds of volume $< V$ will be obtained by Dehn filling on one of the resulting finite collection of manifolds. \square

Theorem 5.5. *There is an algorithm which will determine the finitely many one cusped orientable hyperbolic 3-manifolds N_1, \dots, N_k such that N_i has two exceptional Dehn fillings of distance > 5 .*

Proof. We know that if N is a one-cusped orientable hyperbolic 3-manifold which has an embedded maximal horocusp neighborhood $\mathcal{H} \subset N$ such that $\text{Vol}(\mathcal{H}) > 3$,

then any two Dehn fillings on N have distance ≤ 5 . Thus, we must determine the one-cusped orientable hyperbolic 3-manifolds N which have a maximal cusp of volume ≤ 3 . Such manifolds have a non-free bicuspid subgroup $Q = \langle \alpha, \beta, \gamma \rangle$ of finite index in $\pi_1 N = \Gamma$, such that \mathcal{H} lifts to a maximal cusp of \mathbb{H}^3/Q . Thus, we must determine the bicuspid groups Q which have a maximal cusp of \mathbb{H}^3/Q of volume ≤ 3 . By Theorem 3.2, we know that $\text{Vol}(\mathbb{H}^3/Q) < V(\epsilon)$, for $\epsilon < \pi - 3$. But we don't know how to compute $V(\epsilon)$ explicitly. So we first show that there is an algorithm which will determine all bicuspid groups Q such that \mathbb{H}^3/Q has a maximal cusp of volume $< \pi - \epsilon$.

As in the proof of Lemma 3.1, we normalize a general bicuspid group $Q = \langle \alpha, \beta, \gamma \rangle$ so that

$$\alpha = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \gamma = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix},$$

where $1 \leq |a| \leq |b| \leq 2A/\sqrt{3}$, and $|c| \leq |b|$, where $A = \text{Area}(\partial\mathcal{H})$. This gives a finite parameter space \mathcal{P} to search through, and we know that when $A < 2\pi$, the group Q must either be non-free bicuspid, or else Q is indiscrete. Furthermore, if Q is discrete but not a free product, then either Q has finite covolume, or Q has torsion. In this case, there will be a non-trivial word $w(x, y, z) \in (\langle x \rangle \times \langle y \rangle) * \langle z \rangle \cong (\mathbb{Z} + \mathbb{Z}) * \mathbb{Z}$ such that $w(\alpha, \beta, \gamma) = 1 \in Q$. A lift of $w(\alpha, \beta, \gamma)$ to $\text{SL}_2\mathbb{C}$ gives a matrix $W(a, b, c) = \begin{pmatrix} * & * \\ p(a, b, c) & * \end{pmatrix}$, such that $W(a, b, c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ when the variables (a, b, c) correspond to the group Q . The lower left entry of $W(a, b, c)$ may be regarded as a polynomial $p(a, b, c) \in \mathbb{Z}[a, b, c]$ which vanishes on the parameters corresponding to Q .

If $\mu = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ represents a matrix in $\text{SL}_2\mathbb{C}$, and if $\tilde{\mathcal{H}}$ is the horoball of height 1 centered at ∞ in \mathbb{H}^3 , then $\mu(\tilde{\mathcal{H}})$ is a horoball of height $1/|y|^2$ for $y \neq 0$. Thus, if $\tilde{\mathcal{H}} \cap \mu(\tilde{\mathcal{H}}) = \emptyset$, we see that $|y| \geq 1$. So, if $\mu \in Q$ is an element of the bicuspid group Q , and $\tilde{\mathcal{H}}$ is precisely invariant under Q , then we see that if $|y| < 1$, we must have $\tilde{\mathcal{H}} = \mu(\tilde{\mathcal{H}})$, and therefore μ is a parabolic element, so $y = 0$.

Similarly, if Q is indiscrete, then the closure $\overline{Q} \leq \text{SL}_2\mathbb{C}$ must be dense in a Lie subgroup of $\text{SL}_2\mathbb{C}$, and therefore there are elements $w(\alpha, \beta, \gamma) \in Q$ such that $|W(a, b, c) - I|^2 < \epsilon$, for any small ϵ . In particular, if

$$W(a, b, c) = \begin{pmatrix} * & * \\ p(a, b, c) & * \end{pmatrix},$$

where $p(a, b, c) < 1$, then either $p(a, b, c) = 0$, or the three generator group associated to the parameters (a, b, c) does not have an embedded horocusp $\mathcal{H} \subset \mathbb{H}^3/\langle \alpha, \beta, \gamma \rangle$ which is a projection of a horoball at height 1 in \mathbb{H}^3 , and is therefore not bicuspid with respect to these generators. Find a finite collection of such polynomials $\{p_1, \dots, p_k\} \in \mathbb{Z}[a, b, c]$ so that the sets $\{(a, b, c) | p_i(a, b, c) < 1\}$ cover \mathcal{P} . Each such polynomial is determined by a word $w_i(x, y, z)$. We may find a finite collection since \mathcal{P} is compact. To make this search algorithmic, enumerate $p(a, b, c)$ for all $w(x, y, z) \in \mathbb{Z}^2 * \mathbb{Z}$. Cover \mathcal{P} by compact subsets, such as cubes with dyadic vertices and sidelength 2^{-n} . For a given polynomial $p(a, b, c)$, and a given cube C , one may determine whether $|p(a, b, c)| < 1$ for all points $(a, b, c) \in C$, by determining the maximal value of $p(a, b, c)$ on C , which is algorithmic (an interior maximum is determined by computing a point where the gradient $\nabla p(a, b, c) = 0$, which may be

computed via algebraic geometry, whereas boundary maxima may be determined by Lagrange multipliers inductively on the faces of the cube). Inductively, we refine the covering and increase the number of polynomials, until we find a covering of \mathcal{P} (this is similar to the process employed in [18, Proposition 1.28]).

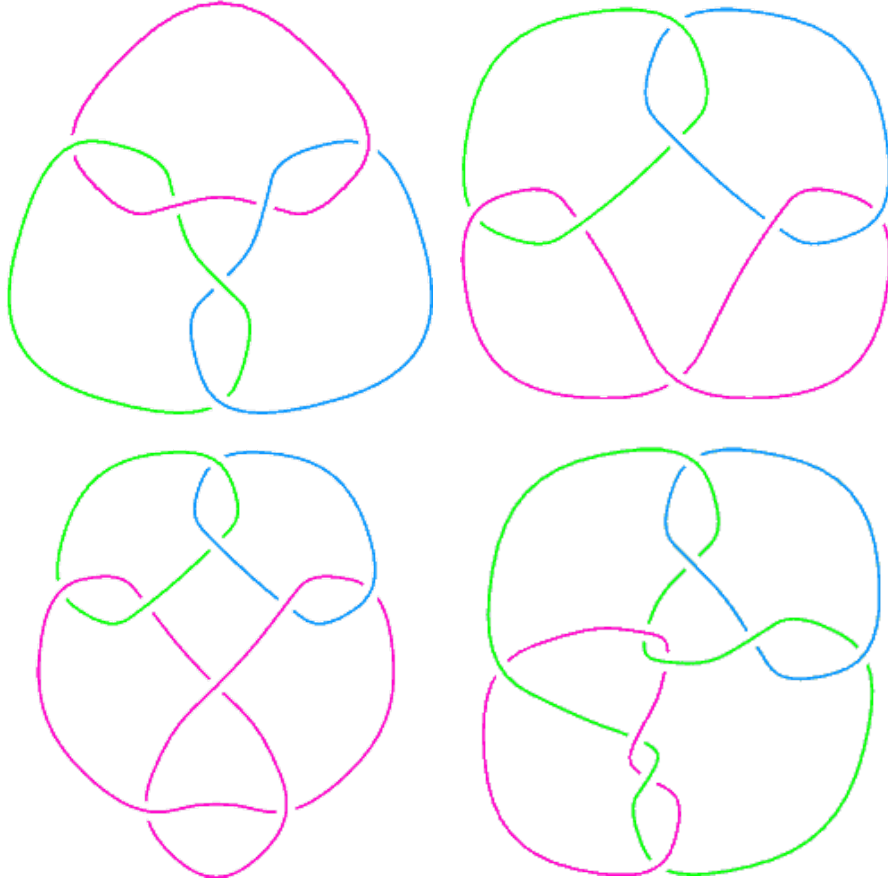
Points $(a, b, c) \in \mathcal{P}$ where $p_i(a, b, c) = 0$ correspond to irreducible bicuspid representations of $\mathbb{Z}^2 * \mathbb{Z}$. If $0 < p_i(a, b, c) < 1$, then either the group is indiscrete, or it does not have a maximal cusp \mathcal{H} normalized as above. Thus, all of the discrete non-free bicuspid groups with a cusp of volume ≤ 3 will occur somewhere in the parameter space \mathcal{P} with some $w_i(\alpha, \beta, \gamma) = 1$, and therefore $p_i(a, b, c) = 0$. By Lemma 5.3, the covolume of the discrete torsion-free Q will be bounded by the complexity $\pi(d(w_i) - 2)$. Let V be the supremum of these numbers. Thus, we have shown the existence of an algorithm to determine a bound on the covolume of bicuspid groups with a maximal cusp of volume ≤ 3 .

The next step is to enumerate all orientable hyperbolic manifolds with volume $< V$. We use Theorem 5.4 to give a finite collection M_1, \dots, M_m of finite volume hyperbolic 3-manifolds so that any hyperbolic 3-manifold of volume $< V$ is obtained by hyperbolic Dehn filling on one of the manifolds M_j . Finally, we use Theorem 5.1 to determine the set of all exceptional Dehn fillings on each M_j . We then search this list of exceptional Dehn fillings for any one cusped manifolds with two exceptional fillings of distance > 5 . By Theorem 4.1, there will be only finitely many isometry types of hyperbolic one cusped manifolds with two exceptional Dehn fillings of distance > 5 . \square

6. CONCLUSION

There are several problems suggested by the results in this paper. Of course, there is Gordon's conjectured classification of one cusped manifolds with two exceptional fillings of distance > 5 (see [24, Problem 1.77, Conjecture (B)]). As we point out, it will suffice to classify the manifolds with a cusp of volume ≤ 3 . Searching through the Snappea census of cusped orientable manifolds [40], there appears to be few one or two-cusped manifolds which have cusp volume ≤ 3 and which are not obtained by Dehn filling on a manifold with corresponding cusp of volume very close to 3 and which have fewer than three cusps. The collection of 3-cusped manifolds with a cusp of volume ≤ 3 is finite. We show several examples of three cusped manifolds with a cusp of volume ≤ 3 in Figure 3, and which give rise to infinite families of 3-manifolds with maximal cusps of volume ≤ 3 via Dehn filling. We remark that if one wants to classify manifolds with > 8 exceptional Dehn fillings, it suffices to classify manifolds with two exceptional Dehn fillings r_1, r_2 with $\Delta(r_1, r_2) \leq 6$. Thus, it suffices to consider manifolds with a cusp of volume $\leq 2\frac{4}{7}$. This would significantly simplify the search space in the algorithm proposed in Theorem 5.5.

A related question is to understand the structure of non-free bicuspid groups. One may obtain examples by considering genus 2 manifolds in which one of the compression bodies C such that $\partial_+ C = \Sigma_2$, $\partial_- C = T^2$. It is natural to ask whether this construction gives all examples. We conjecture that all hyperbolic manifolds with bicuspid fundamental group are genus 2. If so, this would help the search algorithm in Theorem 5.5 since one would need only add relators coming from embedded curves on the boundary of a genus two compression body to obtain all of the bicuspid groups with small cusp volume.

FIGURE 3. Some 3 component links with cusps of volume ≤ 3

It would be nice to go beyond Gordon's conjecture, and to classify all finite-volume orientable hyperbolic manifolds with two exceptional Dehn fillings of distance > 4 . We suspect that the only examples are obtained from Dehn filling on the $(-2, 3, 8)$ pretzel link and the Whitehead link. To achieve this using the methods in this paper, one would have to classify hyperbolic manifolds with cusp volume < 4.5 , and this does not seem feasible. To make further progress using geometric methods, one will likely have to delve further into the geometry of cusped manifolds, and try to understand Ford domains of cusped manifolds with small volume horocusps. For each slope in a cusp, one may consider the minimal area of a complete surface inside the Ford domain which is asymptotic to the given slope. If this area is $> 2\pi$, then the Dehn filling along this slope cannot be exceptional. It may be possible to numerically estimate these areas for a fixed type of Ford domain, and could possibly give better estimates on the number of exceptional surgeries than the estimates obtained from cusp volume. It seems likely that the methods of [17] should be suited to this sort of analysis, and it may be possible to resolve [24, Problem 1.77].

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